Econ 802
Lecture Notes on Chapter 9

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Note: I already did section 9.5 in my notes for chapter 8 so I skip it here. I will also not comment on 9.6, but read it.

This chapter contains an assortment of topics in consumer theory. I will go through them roughly in the same order as Varian.

Endowments in the Budget Constraint (9.1)

The usual utility max problem is $\max_x u(x)$ subject to $px = m$

where $x = (x_1 \dots x_n)$, $p = (p_1 \dots p_n)$

In this situation income (m) is exogenous and independent of the prices

But in some cases (such as general equilibrium models) we want to know where income comes from.
It is often convenient to assume the consumer has a physical endowment of goods denoted by the vector $e = (e_1 \dots e_n)$. Think of these as the quantities of the goods the consumer owns before going to the market and buying/selling things (some of the endowment quantities may be zero).

Note: Varian uses the Greek letter omega (ω) for the endowment vector, but this looks too much like w, which he uses later in the same section to indicate the wage. To avoid possible confusion, I will use e for the endowment vector instead.

Now express income as $m = pe = \sum_i p_i e_i$
This is what the consumer gets by selling their endowment in the market. Once they have income m, they go back to the market and buy a consumption bundle x at the same prices p.

This does not create any problems with utility max. We can still find the Marshallian demands $x(p,m)$ in the usual way, and then substitute $m = pe$ to get $x(p, pe)$.

We will say the consumer is a <u>net supplier</u> of good i if $x_i < e_i$ and a <u>net demander</u> if $x_i > e_i$.

The new issue here is that if a price changes, it affects the demand $x_i(p, pe)$ for good i in two ways: first in the usual way by changing p, and second by changing the value of the endowment pe and therefore changing the income $m = pe$. If we want the <u>total</u> effect of a change in p_j on x_i we need to take both of these into account.

Using some calculus:

$$\frac{dx_i(p,pe)}{dp_j} = \underbrace{\frac{\partial x_i(p,pe)}{\partial p_j}}_{\substack{\text{direct price}\\ \text{effect, holding}\\ \text{income constant}}} + \underbrace{\frac{\partial x_i(p,pe)}{\partial m}\frac{\partial (pe)}{\partial p_j}}_{\substack{\text{indirect price effect}\\ \text{operating through a}\\ \text{change in income}}}$$

Note: This is <u>not</u> the Slutsky equation!

We can, however, substitute the Slutsky equation into the result at the bottom of p. 2:

The Slutsky equation says $\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial m} x_j$

Substitution gives

$$\frac{dx_i}{dp_j} = \underbrace{\frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial m} x_j}_{\text{from Slutsky}} + \underbrace{\frac{\partial x_i}{\partial m} e_j}_{\text{price effect operating through change in income}}$$

Therefore

$$\frac{dx_i}{dp_j} = \underbrace{\frac{\partial h_i}{\partial p_j}}_{\text{the usual substitution effect}} + \underbrace{\frac{\partial x_i}{\partial m}(e_j - x_j)}_{\text{an "income effect" that depends on whether the consumer is a net supplier } (e_j > x_j) \text{ or a net demander } (e_j < x_j) \text{ of good } j}$$

So for instance if $i = j$ and we have a normal good and the consumer is a net demander ($x_i > e_i$) then both terms are negative and $\frac{dx_i}{dp_i} < 0$.

But if the consumer is a net supplier ($x_i < e_i$) then the terms have opposite signs and the result is unclear ($p_i \uparrow$ implies $x_i \downarrow$ due to substitution effect but $x_i \uparrow$ due to higher income)

Note: There is nothing here about inferior goods or Giffen goods. The ambiguity arises for a normal good because $p_i \uparrow$ may increase the consumer's income, which means x_i tends to rise instead of falling.

The most common application of endowments in the budget constraint involves labor-leisure tradeoffs. Let's see how this works.

Varian sets this up in several different ways, and he changes the meaning of m from total income to "non-labor" income, which is confusing. My notation will be a little different.

Assume total time is T. This can be divided between work hours (H) and leisure hours (L) with $H + L = T$.

[This is a scalar]

The consumer cares about two goods: consumption (c) and leisure (L). The goal is to maximize the utility function $u(c, L)$. Note that the consumption set here would have $0 \leq L \leq T$ because there is a physical upper bound on leisure.

The budget constraint is $pc \leq \underbrace{w(T-L)}_{\text{labor income}} + \underbrace{r}_{\text{non-labor income}}$ (where w is the wage)

Rearranging this gives

$$pc + wL \leq wT + r$$

I will use $m = wT + r$.

Varian uses m where I use r, which is confusing.

Note: you can imagine that the consumer first sells off their entire time endowment T at the wage rate w, and then "purchases" some leisure L at the price w. Obviously this is not a good physical description but it might help in understanding the economics.

Now write the problem as $\max u(c, L)$
subject to $pc + wL \le m$

This takes us back to the usual problem of solving for the Marshallian demands for c and L as functions of (p, w, m). There is no new issue with this.

Once you have the Marshallian demands $\begin{cases} c(p, w, m) \\ L(p, w, m) \end{cases}$ just substitute $m = wT + r$ to get

$$\begin{cases} c[p, w, wT + r] \\ L[p, w, wT + r] \end{cases}$$

Suppose we want to know what happens to leisure as the wage rate w increases. The total effect is

$$\frac{dL}{dw} = \frac{\partial L}{\partial w} + \frac{\partial L}{\partial m}\frac{\partial m}{\partial w} \qquad \left(m = wT + r \text{ so } \frac{\partial m}{\partial w} = T\right)$$

According to Slutsky,

$$\frac{\partial L}{\partial w} = \underbrace{\frac{\partial h_L}{\partial w}}_{\text{subs effect}} - \underbrace{\frac{\partial L}{\partial m} \cdot L}_{\text{income effect}} \qquad \left(h_L \text{ is the Hicksian demand for leisure}\right)$$

Substitute this into the previous equation to get

$$\frac{dL}{dw} = \underbrace{\frac{\partial h_L}{\partial w}}_{\text{negative subs effect}} + \frac{\partial L}{\partial m}\underbrace{\left[T - L(p, w, m)\right]}_{\substack{= \text{ work hours } (H) \\ \text{so positive if } H > 0.}}.$$

We usually think leisure is a normal good ($\frac{\partial L}{\partial m} > 0$ so if income goes up, you don't work so much)
But this implies a negative substitution effect and a positive income effect (due to a normal good and the fact that the consumer is a net supplier of time to the labor market)

Therefore the overall sign of $\frac{dL}{dw}$ is ambiguous.

It would not be surprising to have $\frac{dL}{dw} \approx 0$

so we could observe a very low elasticity for leisure as a function of the wage (maybe zero, maybe negative).

The same is true of work hours because due to the time constraint $H + L = T$, we must have

$$\frac{\partial H}{\partial w} + \frac{\partial L}{\partial w} = 0$$

so if $\frac{\partial L}{\partial w}$ is close to zero, $\frac{\partial H}{\partial w}$ is also close to zero.

Thus the observed elasticity of labor supply with respect to the wage could be approximately zero (maybe positive, maybe negative)

This has nothing to do with inferior goods or Giffen goods. It is just that a price change is pushing in one direction while an income change is pushing in the other direction.

I will skip over section 9.2 on homothetic functions, although you should read it.

The next topic I will deal with is "aggregation across goods" (section 9.3). This is also called separability.

<u>Separability</u>

Suppose we know how much money a consumer spends on food in general. Can we describe the allocation of this part of income among particular food items (spinach, macaroni, etc) without knowing the details of expenditures for non-food items?

This kind of question often arises when people are estimating demand functions. Is it necessary to estimate the demands for all goods as a function of all prices (and total income)? Or can we estimate demands for a subset of goods using only the prices for that subset, and expenditure on that subset?

There are two types of separability that can be useful in thinking about such questions.

1. <u>Hicksian separability</u>. This imposes restrictions on the nature of price changes, but makes no assumption about the structure of preferences.

2. <u>Functional separability</u>. This makes no assumption about the nature of price changes, but it imposes a special structure on the utility function.

<u>Hicksian separability</u> (note: This has no connection with the idea of Hicksian demand functions, other than the fact that the name of the same economist is attached to both)

This framework provides a justification for the common practice of drawing a graph with a particular good on the horizontal axis and "all other goods" on the vertical axis — under certain conditions it is valid to aggregate all of the other goods.

Divide the consumption vector in the form (x, z) with a corresponding price vector (p, q). For concreteness I will think of x as a vector of food items and z as various non-food goods.

Assume $q = t q_0$ for some fixed vector q_0, with $t > 0$. This says that relative prices within q do not change. Or: price movements for the z commodities leave the ratios in q_0 unchanged.

We can write indirect utility as

$$v(p, t, m) = \max_{(x,z)} u(x, z) \quad \text{subject to } px + t q_0 z = m$$

because only t ever changes for the z goods. Think of $(q_0 z)$ as a composite commodity whose price is t. In fact, we can define $Z = q_0 z$ ~~and~~ and use the trick of going from an indirect utility function back to a direct utility function. This gives

$$w(x, Z) = \min_{(p,\pi)} v(p, \pi, 1) \text{ subject to } px + \pi Z = 1$$

So it is just as if the consumer is maximizing the direct utility function $w(x, Z)$ defined over food items and "all other goods" (Z), where π is the price of Z.

You can think of π as a price index. Write the Marshallian demands ~~as $x(p, \pi)$~~ for the food items as

$$x(p, \pi, m) = x\left(\frac{p}{\pi}, 1, \frac{m}{\pi}\right)$$

Where we have used the zero degree homogeneity of the Marshallian demands and multiplied through by $\frac{1}{\pi}$. This says that the demands for individual food items depend on "real prices" $\frac{p}{\pi}$ and "real income" $\frac{m}{\pi}$. However keep in mind that π is a price index for the non-food goods (not all goods).

The Hicksian approach is valid as long as it is reasonable to assume zero or "small" changes in the relative prices of the non-food goods (or more generally, "all other goods," meaning the goods you don't want to think about, maybe because you don't have detailed data about them).

Functional seperability

Suppose we don't want to impose restrictions on the ways in which prices can change. An alternative is to make assumptions about the nature of preferences.

Assume the following is true:

$$(x, z) \succ (x', z) \text{ if and only if } (x, z'') \succ (x', z'')$$

where this is true for all x, x' and z, z''.

We are using the preference notation from Chapter 7 here. What this says in words is that if the consumer prefers x to x' when the non-food goods are z, the consumer also prefers x to x' when the non-food goods are z''. Or to put it another way: preferences between two bundles of food items (x and x') are independent of what bundle of non-food items (z or z'') the consumer happens to have.

With this preference assumption plus local non-satiation, we can write the utility function in the special form

$u[w(x), z]$ where u is increasing in w.

As before, we write consumption bundles as (x, z) and the corresponding price vectors as (p, q).

Note: Varian uses $v(x)$ in place of $w(x)$ (see p. 150). This is confusing because we would normally use v for an indirect utility function. Therefore I will use $w(x)$ instead.

In general, we would write the Marshallian demands as $x(p, q, m)$ and $z(p, q, m)$ because demands depend on all of the prices.

BUT: if we know the total expenditure on the x goods, we can compute demands for the individual x goods without simultaneously solving for the z demands.

Write expenditure on the x goods as $m^x = p \cdot x(p, q, m)$. Then solve the problem

$$\max_x \; w(x) \text{ subject to } p \cdot x = m^x$$

Clearly it is necessary to solve this problem in order to maximize overall utility, because u is increasing in w. This allows us to solve for Marshallian demands of the form

$x(p, m^x)$ where the demands for individual food items are conditional on m^x and do not directly depend on the non-food prices q.

This is nice, however, to determine m^x we still generally need to know all the prices (p, q) and the overall income m because $m^x = p \cdot x(p, q, m)$. So it may appear that we haven't really accomplished anything. Two points on this:

(i) You may know m^x from empirical data (it would be sufficient to observe the p and x vectors). Then if you are willing to make an assumption of functional separability, you can estimate the "conditional" demands $x(p, m^x)$, without knowing q, z, or m. Note: this would be theoretically valid. However, you might want to ask an econometrician about whether there is a problem associated with the endogeneity of m^x.

② $w(x)$ may be homothetic (see section 9.2)
If it is we can write $w(x) = h[g(x)]$ where $h' > 0$ and $g(x)$ is linearly homogeneous.
Notice that h is irrelevant because we have ordinal utility and the h function can be absorbed into the u function. Thus we can just assume that $w(x)$ is linearly homogeneous, which is simpler.

This is like what happens with a cost function when we have constant returns (m^x enters multiplicatively)

In this situation, the indirect utility function can be written as $v(p)m^x = \max_x w(x)$ subject to $px = m^x$.

Substitute this for $w(x)$ in the overall utility function to get

$$\max_{m^x, z} u[v(p)m^x, z] \text{ subject to } m^x + qz = m$$

This allows you to solve for m^x without thinking about the details of the p vector, and just using $v(p)$ instead.

Another interpretation: define a composite food commodity $X = v(p)m^x$. Then write the consumer's overall problem as

$$\max_{X, z} u[X, z] \text{ subject to } X\frac{1}{v(p)} + qz = m$$

where $\frac{1}{v(p)}$ serves as the "price" of the composite good X.

Now we have a two stage problem:
① compute the optimal z and m^x (it is not necessary to solve for $x(p,q,m)$ at this stage)
② given m^x, solve for $x(p, m^x)$. This part is easy because once the proportions of the x goods are known, we can use homotheticity to scale x up and down as a function of the expenditure m^x.

Aggregating Across Consumers (Section 9.4)

If there are n consumers ($i = 1 \ldots n$), the market (aggregate) demand for good k is

$$X^k(p, m_1, \ldots m_n) = \sum_{i=1}^{n} x_i^k(p, m_i)$$

where $x_i^k(p, m_i)$ is consumer i's Marshallian demand for good k and m_i is consumer i's income.
We can write the market (aggregate) demands in vector form as

$$X(p, m_1, \ldots m_n) = \sum_{i=1}^{n} x_i(p, m_i)$$

What can we say about the properties of the market demand functions? In general, not much.

1. If the individual consumers all have continuous demand functions, then the aggregate demands will also be continuous.
2. We have zero degree homogeneity in $(p, m_1, \ldots m_n)$ due to the zero homogeneity of the individual demands
3. There is an "adding up" restriction:

$$\sum_k p_k X^k(p, m_1, \ldots m_n) = \sum_i m_i = M$$

where M is total income (this follows from the budget constraints.

Beyond this, we can't say anything. At the aggregate level, there is no Slutsky equation, no symmetry, no negative semidefiniteness, etc. We cannot say that Hicksian "aggregate" demands slope down, because the aggregate demands are not derived from an aggregate utility function.

It would be nice if we could simplify the aggregate demands in the form $X(p, M)$ so they only depend on total income M, but in general this won't work - the aggregate demands will depend on how income is distributed among the individual consumers.

This raises the following question: could you ever find a utility function such that the market demands are the outcome of maximizing this function? ie can we think of $X(p,M)$ as maximizing some $U(X)$ subject to $p \cdot X = M$?

It turns out that this is possible in one case. Suppose all consumers have indirect utility functions of the Gorman form:

$$v_i(p, m_i) = a_i(p) + b(p)\, m_i$$

(notice that $a_i(p)$ can have an i subscript but $b(p)$ must be identical for everyone)

In this case the Marshallian demand for good k by consumer i is

$$x_i^k(p, m_i) = -\frac{\dfrac{\partial v_i(p, m_i)}{\partial p_k}}{\dfrac{\partial v_i(p, m_i)}{\partial m_i}} = -\frac{\left[\dfrac{\partial a_i(p)}{\partial p_k} + \dfrac{\partial b(p)}{\partial p_k} m_i\right]}{b(p)}$$

(use Roy's Identity)

Now sum over all consumers:

$$X^k(p, m_1 \ldots m_n) = -\frac{1}{b(p)}\left[\sum_i \frac{\partial a_i(p)}{\partial p_k} + \frac{\partial b(p)}{\partial p_k} \underbrace{\sum_i m_i}_{= M}\right]$$

so $X^k(p, M)$ depends only on total income M, not how it is distributed.

Is there an "aggregate" indirect utility function that would generate (or "rationalize") these aggregate demands?

Yes: define $V(p, M) \equiv \sum_{i=1}^{n} a_i(p) + b(p) M$

Use Roy's identity to compute the Marshallian demands generated by this indirect utility function:

$$X^k(p, M) = -\frac{\frac{\partial V(p,M)}{\partial p_k}}{\frac{\partial V(p,M)}{\partial M}} = -\frac{\left[\sum_{i=1}^{n} \frac{\partial a_i(p)}{\partial p_k} + \frac{\partial b(p)}{\partial p_k} M\right]}{b(p)}$$

You can verify that these aggregate demands are identical to the ones we obtained from optimization by the individual consumers.

This is both necessary and sufficient; the Gorman form is the most general indirect utility function which allows aggregation across consumers in this way. (We get a representative consumer having an indirect utility function of the same form)

When we can aggregate in this way, all of the usual properties from individual consumer behavior carry over to the market level: there is an aggregate Slutsky equation, we get the usual symmetry and negative semi-definiteness at the aggregate level, there are Hicksian demand curves that slope down, and so on.

How restrictive is the Gorman form?

(1) Varian says on p. 154 that this works if you have a homothetic utility function. But you need to be careful about this. Suppose consumer i has a homothetic utility function $u_i(x_i)$. In fact just assume it is linearly homogeneous because this represents the same preferences and is simpler. Then i has an indirect utility function

$$v_i(p, m_i) = b_i(p) m_i$$

which is in the Gorman form.

<u>But</u> we would need $b_i(p) = b(p)$ for all i in order to aggregate across consumers, and this may or may not be true. It is like having Cobb-Douglas production functions with CRS; if the exponents differ across firms, $b_i(p)$ will vary across i.

(2) Varian also says that aggregation works for quasi-linear utility functions. This is correct. Suppose there are two goods and consumer i has the direct utility function $u_i(x_i) = z_i(x_{i1}) + x_{i2}$ where z_i is some increasing and strictly concave function. Then

$$v_i(p, m_i) = \max z_i(x_{i1}) + x_{i2} \text{ s.t. } p_1 x_{i1} + p_2 x_{i2} = m_i$$

$$= \max z(x_{i1}) + \frac{m_i - p_1 x_{i1}}{p_2}$$

FOC: $z_i'(x_{i1}) = \frac{p_1}{p_2}$

(This does not depend on income)

Let $x_{i1}(p)$ be the Marshallian demand from FOC

Then $v_i(p, m_i) = \underbrace{z_i[x_{i1}(p)] - \frac{p_1}{p_2} x_{i1}(p)}_{\text{call this } a_i(p)} + \frac{m_i}{p_2}$

Again, this is in the Gorman form at the individual level. and write $b(p) = \frac{1}{p_2}$ (same for all i) <u>AND</u> $b(p) = \frac{1}{p_2}$ is the same for all i, so aggregation works and there is an aggregate utility function in the Gorman form.